Analysis I

Teacher: Leonid Monin

Lectures: Mondays, Wednesdays

10:15 – 12:00

In room CM 1105

Also streamed on zoom.

SOME MATERIALS:

- Notes availible on Moodle
- Recordings of Lectures
- Hand written notes from lectures.

Exercises

How it works?

- Exercise sheets posted
- You work on them by yourself
- Assistants will help you during exercise sessions

Mondays 8:15-10:00 in BS160, BS170

REVIEWED EXERCISES

There will be 2-3 special exercise sheets which you could hand in and they will be checked.

- This is for you to check your knowledge
  No effect to the final mark

EXAM:

- There will be MCQs, TF part and open part.

- On week 9 there will be a practice exam.

WHAT IS THIS COURSE ABOUT?

- Real and complex numbers
- Sequences and their limits
- Series ← infinite sums $a_1 + a_2 + a_3 + \dots$
- Theory of functions:

  Operations with functions, continuity,

  derivatives, Integration, ...

WHAT IS THIS COURSE ABOUT?

We will learn mathematics rigorously:

Main objects:

Definitions

Theorems (or Lemmas or Propositions...)

Proofs

Mathematical proof:

We will mostly see two types:

- **Constructive proof:**

Conclusion

Hypothesis

$$A \Rightarrow B \Rightarrow C \Rightarrow D$$

- **Proof by contradiction:**

Goal: show that A is true.
Assume that A is False then show that this leads to contradiction.

Example Theorem $\sqrt{3}$ is not a rational number.

Recall, a rational number is a fraction $\frac{a}{b}$ with a, b natural numbers and $b \neq 0$

Moreover, we can assume that

a, b are coprime:

$$\gcd(a, b) = 1.$$

Theorem $\sqrt{3}$ is not a rational number.

Proof (by contradiction)

1st Assume that $\sqrt{3}$ is rational

$$\sqrt{3} = \frac{a}{b} \text{ with } \gcd(a, b) = 1$$

$$\sqrt{3} = \frac{a}{b} \Rightarrow (\sqrt{3})^2 = \left(\frac{a}{b}\right)^2$$

$$\Rightarrow 3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2$$

$\Rightarrow$ a is divisible by 3

In other words, there exists an integer c s.t. $a = 3 \cdot c$

$$\underbrace{3b^2 = a^2 \quad \text{and} \quad a = 3c}$$

$$\Downarrow$$

$$3b^2 = (3c)^2 = 9 \cdot c^2$$

$$\Downarrow$$

$$b^2 = 3 \cdot c^2 \Rightarrow$$ b is divisible by 3

So we assumed that

$$\sqrt{3} = \frac{a}{b} \quad \text{with} \quad \gcd(a,b) = 1$$

we showed that this assumption implyes $3 \mid b$, $3 \mid a \Rightarrow \gcd(a,b) \geq 3$

$\rightsquigarrow$ contradiction $\blacksquare$

Notation ($x \mid y$ means y is divisible by x)

Remark the proof shows

that $\sqrt{p}$ is not

rational for any prime p.

$a = p_1 \cdot p_2 \dots p_k$ with p_i - prime

$a^2 = (p_1 \dots p_k)(p_1 \dots p_k)$

0. Preliminaries

Sets

A set is a collection of objects called elements

there are 2 ways to define sets:

1. List elements of your set

$$A = \{ 1, 5, 3, 7 \}$$

elements could be anything!

e.g - pairs of numbers (1, 2)
or sequences

- symbols
- sets

2. You can define set by listing properties of its elements

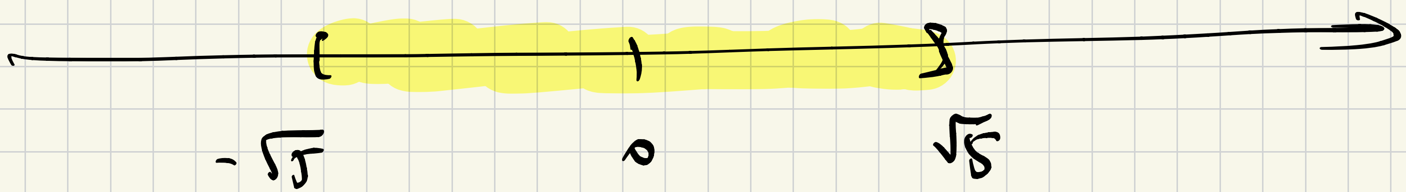
Example: $A = \left\{ a \;\middle|\; \begin{array}{l} a \text{ is integer} \\ a \text{ is divisible by } 3 \end{array} \right\}$

Notation such that

$= \{ \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \}$

$$B = \left\{ x \;\middle|\; \begin{array}{l} x^2 \le 5 \\ x \text{ real number} \end{array} \right\}$$

such that



More on sets

Notation / Definitions

A, B are two sets,

- We write $a \in A$ if a is an element of A

  "belongs to"

- We write $a \notin A$ if a is Not an element of A

<u>Example</u> $X = \{1, 2, d\}$

then $1 \in X$, $2 \in X$, $d \in X$

$5 \notin X$, $k \notin X$ ..

- We write $A \subseteq B$ "is a subset of" if any element of A is also an element of B.
- We write $A \nsubseteq B$ if there exists element $a \in A$ such that $a \notin B$.

Example

$$X = \{1, 2, d\}$$

$$Y = \{1, 2\}, \quad Z = \{1, 5\}$$

Then

$Y \subsetneq X$ since both $1 \in X$ and $2 \in X$

and $Y \neq X$

$Z \not\subset X$ since $5 \in Z$, but $5 \notin X$

Alternative notations:

$A \subset B$ equivalent to

$A \subseteq B$

$A \subsetneq B$ A is subset of B

but $A \neq B$

$A = B$ if they have
"equal" the same elements.

I.e., if any $a \in A$ is also element of B $A \subset B$

$B \subset A$ and any element of B is an element of A

Remark

$$\boxed{\begin{gathered} A = B \\ \Updownarrow \\ A \subset B \text{ and } B \subset A \end{gathered}}$$

Notation/Definition (Empty set).

$$\emptyset = \{ \ \}$$

Remark For any set A

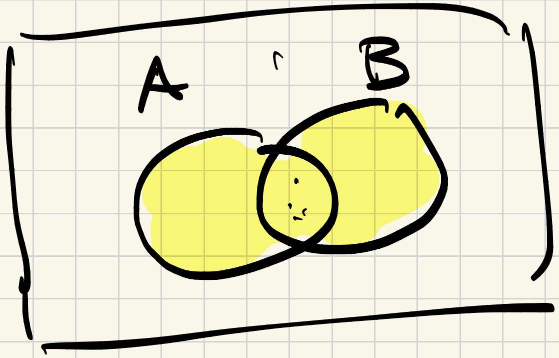
$$\emptyset \subset A.$$

In particular, $\emptyset \subset \emptyset$.

Operations with sets

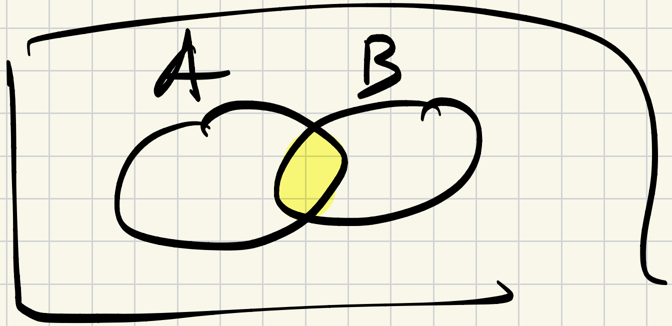
Union.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



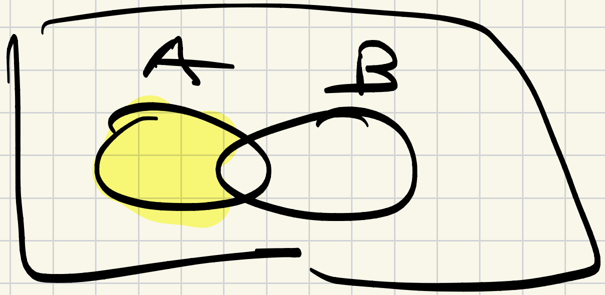
Intersection

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



Complement

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$



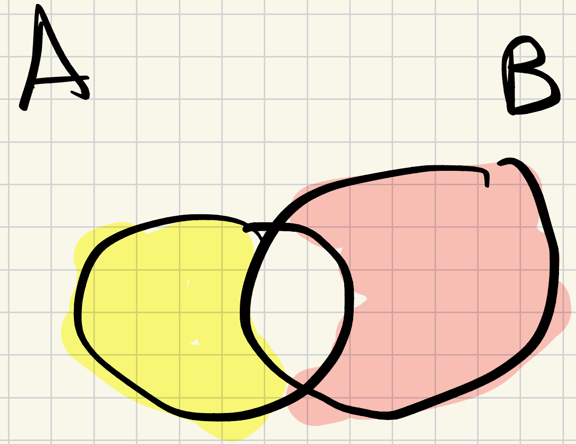
Example:

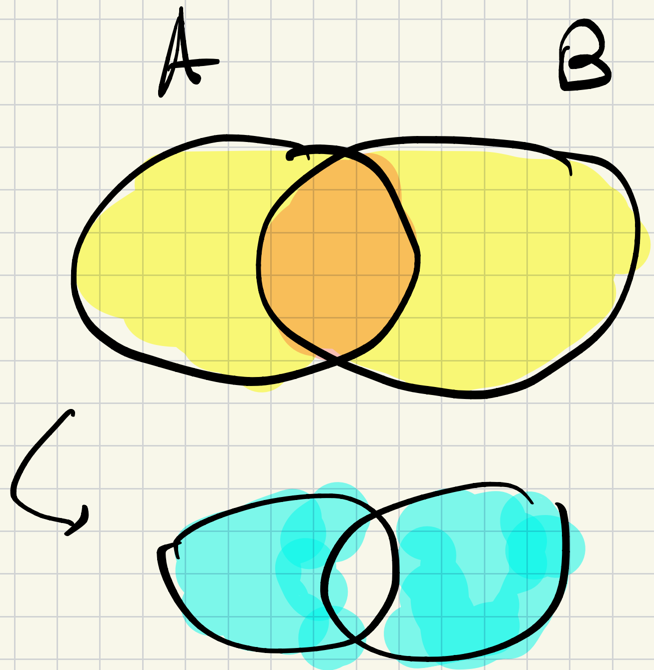
Symmetric difference:

$$(A \setminus B) \cup (B \setminus A)$$

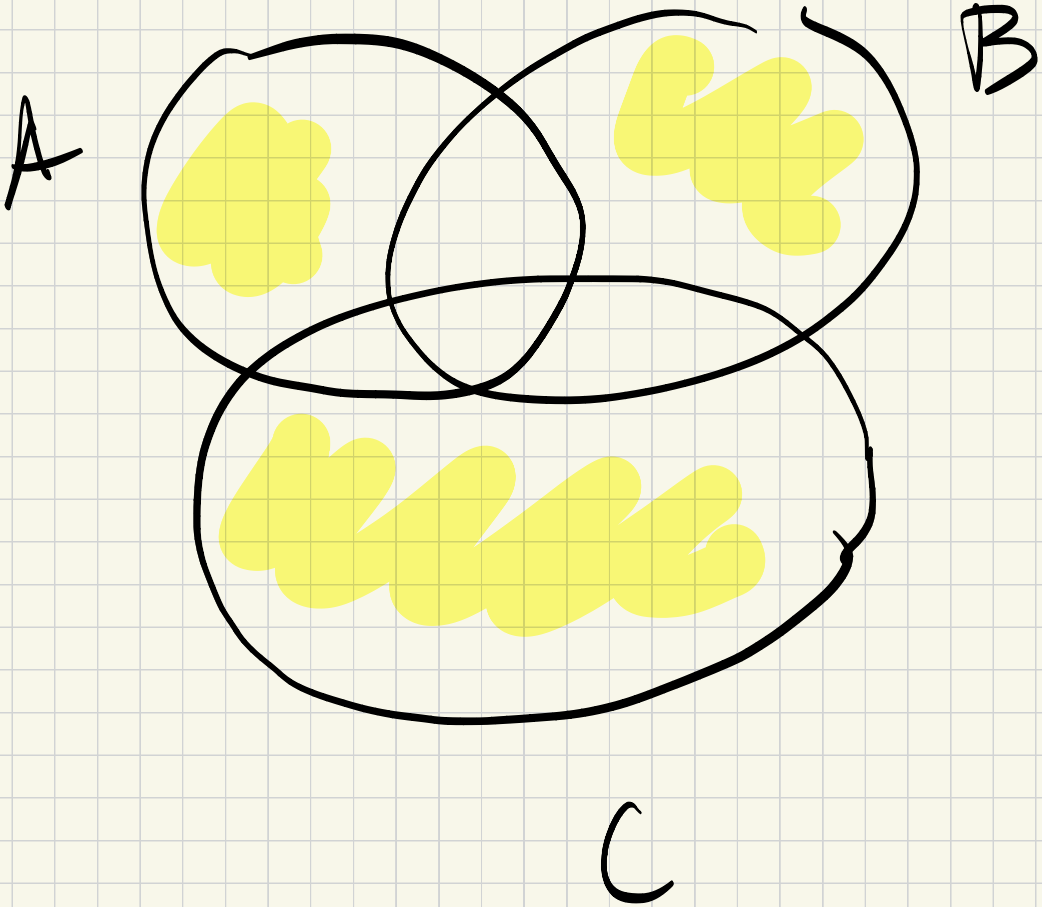
$$\|$$

$$(A \cup B) \setminus (A \cap B)$$





One notation is $A \,\Delta\, B$



Exercise. write a set operation which defines dashed region.

Cartesian product

$$A \times B := \left\{ (a,b) \;\middle|\; \begin{array}{l} a \in A \\ b \in B \end{array} \right\}$$

Remark pair is ordered:

$$(a,b) \neq (b,a)$$

Example $A = \{2, 3, 5\}$, $B = \{1, 6\}$

$$A \times B = \{ (2,1), (3,1), (5,1), (2,6), (3,6), (5,6) \}$$

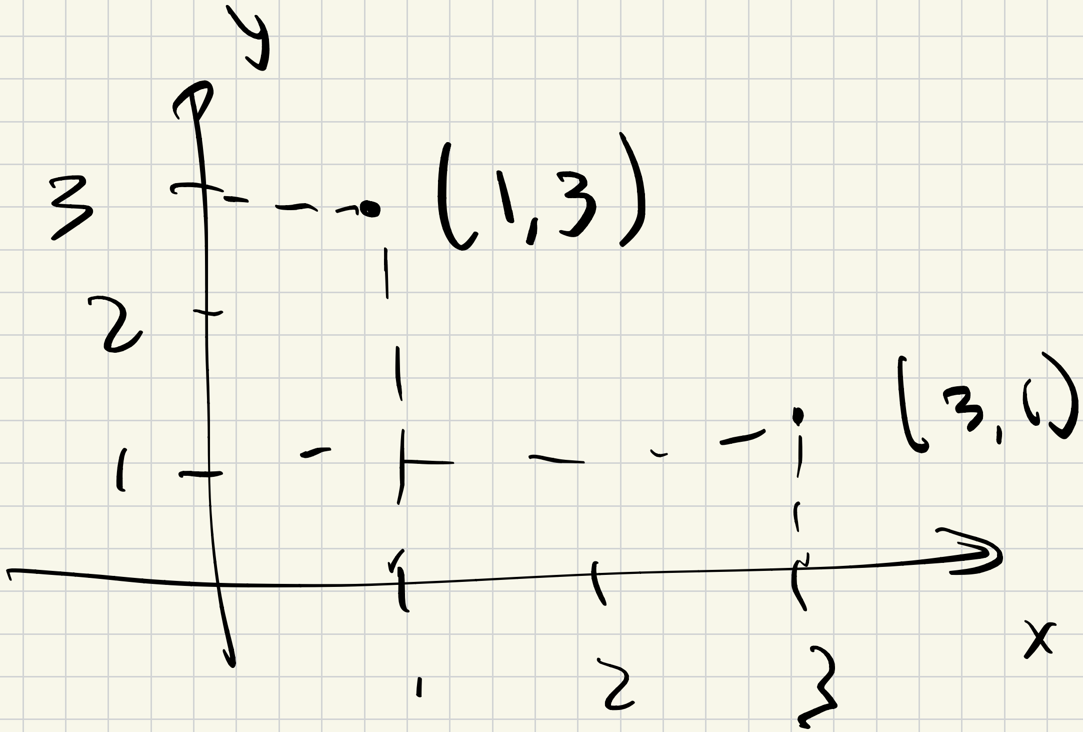
Remark It lets A a b

<u>Remark</u> If A, B are finite sets, denote by $|A|$ and $|B|$ the number of elements in A and B, then

$$|A \times B| = |A| \cdot |B| .$$

Example

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left\{ (a, b) \;\middle|\; \begin{array}{c} a, b \\ \text{real numbers} \end{array} \right\}$$



Some standard sets of numbers:

$\mathbb{N} = \{0, 1, 2, \dots\}$ - non-negative integers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ all integers

$\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$

$\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b - \text{integers} \quad b \in \mathbb{Z}^* \right\}$$

$b \neq 0$

$$\mathbb{Q}^* = \underbrace{\mathbb{Q} \setminus \{0\}}_{\text{non-zero rational numbers}}$$